

DETECTING AND REALISING CHARACTERISTIC CLASSES OF MANIFOLD BUNDLES

SØREN GALATIUS AND OSCAR RANDAL-WILLIAMS

ABSTRACT. We apply our earlier work on the higher-dimensional analogue of the Mumford conjecture to two questions. Inspired by work of Ebert ([Ebe09], [Ebe11]) we prove non-triviality of certain characteristic classes of bundles of smooth closed manifolds. Inspired by work of Church–Farb–Thibault and Church–Crossley–Giansiracusa ([CFT12], [CCG12]) we investigate the dependence of characteristic classes of bundles on characteristic numbers of its fibre, total space and base space.

1. INTRODUCTION AND STATEMENT OF RESULTS

A smooth bundle of closed oriented d -manifolds $\pi : E \rightarrow B$ is a proper submersion with d -dimensional fibres, together with an orientation of the *vertical tangent bundle* $T_\pi E = \text{Ker}(D\pi)$. A characteristic class of such bundles associates to this data an element in $H^*(B)$ which is natural in the bundle. A useful way to define such classes goes via the *parametrised Pontryagin–Thom construction*, which to each $\pi : E \rightarrow B$ associates a map

$$\overline{\alpha}_\pi : B \longrightarrow \Omega_0^\infty \mathbf{MTSO}(d),$$

well defined up to homotopy and natural with respect to pull back of bundles. We recall the definition of the space $\Omega_0^\infty \mathbf{MTSO}(d)$ and the construction of $\overline{\alpha}_\pi$ below. Each cohomology class $c \in H^*(\Omega_0^\infty \mathbf{MTSO}(d))$ gives a characteristic class $\overline{\alpha}_\pi^*(c) \in H^*(B)$, but it is far from clear whether these characteristic classes are non-trivial. This is the *detection question*, i.e. if $c \neq 0$, does there exist a bundle $\pi : E \rightarrow B$ with $\overline{\alpha}_\pi^*(c) \neq 0 \in H^*(B)$? When cohomology is taken with rational coefficients, Ebert ([Ebe11], [Ebe09]) has obtained a complete answer for all d . (He proves that the answer is “yes” when d is even and “no” when d is odd.) In this paper we consider the detection question with arbitrary coefficients. We shall give an affirmative answer when d is an even number greater than 4, and also obtain more control over the detecting bundles. The following is our main result in this direction.

Theorem 1.1. *Let $2n \neq 4$ and $f \in \Omega_{2n}^{SO}$ be a bordism class. Let k be an abelian group and $c \in H^*(\Omega_0^\infty \mathbf{MTSO}(2n); k)$ be a non-zero class. Then there exists a smooth bundle of closed oriented $2n$ -manifolds $\pi : E \rightarrow B$ such that*

- (i) $\overline{\alpha}_\pi^*(c) \neq 0 \in H^*(B; k)$,
- (ii) the fibres of π lie in the bordism class f ,
- (iii) there is a manifold M with boundary, such that $(M, \partial M)$ is $(n-1)$ -connected and the Gauss map $M \rightarrow BSO$ is n -connected, with a fibrewise embedding of a trivial subbundle $B \times M \subset E$ such that $(E, B \times M)$ is $(n-1)$ -connected.

2010 *Mathematics Subject Classification.* 57R22, 57R65, 57R20, 55P47.

S. Galatius was partially supported by NSF grant DMS-1105058, and both authors were supported by ERC Advanced Grant No. 228082, and the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.

For $k = \mathbb{Q}$, Theorem 1.1 reproduces Ebert’s result, but with an entirely different proof. Ebert’s argument used the known structure of $H^*(\Omega_0^\infty \mathbf{MTSO}(2n); \mathbb{Q})$ as a Hopf algebra to reduce his detection result to certain explicit characteristic classes $\kappa_c \in H^p(\Omega_0^\infty \mathbf{MTSO}(2n); \mathbb{Q})$, the “generalised Miller–Morita–Mumford classes”, associated to a monomial $c \in H^{p+2n}(BSO(2n))$ in the Euler class and Pontryagin classes (we will describe these κ_c below). He then uses an explicit construction of smooth bundles to detect these classes. We will deduce Theorem 1.1 as an application of the higher-dimensional version of the Madsen–Weiss theorem from our previous paper [GRW12].

By Theorem 1.1, characteristic classes can be detected using bundles whose fibre lies in a prescribed bordism class. We also investigate the question of detecting classes when the bordism class of the total space and base space is prescribed. Before describing our results in this direction, we recall some definitions.

1.1. The parametrised Pontryagin–Thom construction. We recall the definition of the space $\Omega_0^\infty \mathbf{MTSO}(d)$, and the map $\bar{\alpha}_\pi : B \rightarrow \Omega_0^\infty \mathbf{MTSO}(d)$ associated to a smooth bundle of closed oriented d -manifolds $\pi : E \rightarrow B$.

Let γ_N^\perp denote the $(N - d)$ -dimensional bundle over the oriented Grassmannian $\mathrm{Gr}_d^+(\mathbb{R}^N)$ and let $\mathrm{Th}(\gamma_N^\perp)$ denote its Thom space. There is a natural map $S^1 \wedge \mathrm{Th}(\gamma_N^\perp) \rightarrow \mathrm{Th}(\gamma_{N+1}^\perp)$, and we define the infinite loop space

$$\Omega^\infty \mathbf{MTSO}(d) = \operatorname{colim}_{N \rightarrow \infty} \Omega^N \mathrm{Th}(\gamma_N^\perp).$$

Given a smooth bundle of closed oriented d -manifolds $\pi : E \rightarrow B$, we may pick an embedding $j : E \rightarrow B \times \mathbb{R}^N$ over B , and extend to an open embedding of the fibrewise normal bundle $\nu E \rightarrow B \times \mathbb{R}^N$. Then the Pontryagin–Thom collapse construction gives a based map $B_+ \wedge S^N \rightarrow \mathrm{Th}(\nu)$. The embedding j also induces a map $\tau : E \rightarrow \mathrm{Gr}_d^+(\mathbb{R}^N)$ with $\tau^*(\gamma_N^\perp) = \nu$, and we have a composition $B_+ \wedge S^N \rightarrow \mathrm{Th}(\nu) \rightarrow \mathrm{Th}(\gamma_N^\perp)$, whose adjoint gives a map

$$\alpha_\pi : B \rightarrow \Omega^N \mathrm{Th}(\gamma_N^\perp) \subset \Omega^\infty \mathbf{MTSO}(d).$$

For large N , the embedding $j : E \rightarrow B \times \mathbb{R}^N$ is unique up to isotopy, so the map $\alpha_\pi : B \rightarrow \Omega^\infty \mathbf{MTSO}(d)$ has well defined homotopy class, depending only on the bundle $\pi : E \rightarrow B$. All path components of $\Omega^\infty \mathbf{MTSO}(d)$ are homotopy equivalent, and we let $\bar{\alpha}_\pi : B \rightarrow \Omega_0^\infty \mathbf{MTSO}(d)$ denote the translation to the basepoint component.

Let us now recall the definition of the *generalised Miller–Morita–Mumford classes*. These are universal classes

$$\kappa_c \in H^p(\Omega^\infty \mathbf{MTSO}(d))$$

associated to a class $c \in H^{p+d}(BSO(d))$ by applying the composition

$$H^{p+d}(BSO(d)) \xrightarrow{\mathrm{Thom}} H^p(\mathbf{MTSO}(d)) \xrightarrow{\sigma} H^p(\Omega^\infty \mathbf{MTSO}(d)),$$

where the first map is the Thom isomorphism, and the second map is the cohomology suspension. Given a bundle $\pi : E \rightarrow B$ with corresponding Pontryagin–Thom map $\bar{\alpha}_\pi$, we obtain the equation

$$\bar{\alpha}_\pi^*(\kappa_c) = \pi_!(c(T_\pi E)) \in H^p(B),$$

where $c(T_\pi E) \in H^{p+d}(E)$ denotes the characteristic class of oriented vector bundles c applied to the bundle $T_\pi E$, and $\pi_! : H^{p+d}(E) \rightarrow H^p(B)$ denotes the Gysin map (fibrewise integration).

Finally let us recall the rational cohomology of $\Omega_0^\infty \mathbf{MTSO}(d)$, in the case where d is even, say $d = 2n$. To this end, let $\mathcal{B} \subset H^*(BSO(2n); \mathbb{Q})$ be the set of monomials

in the Euler class and the Pontryagin classes whose total degree is greater than $2n$. Then the natural map induces an isomorphism

$$\mathbb{Q}[\kappa_c \mid c \in \mathcal{B}] \xrightarrow{\cong} H^*(\Omega_0^\infty \text{MTSO}(2n); \mathbb{Q}).$$

1.2. Bundles with prescribed characteristic numbers. If $\pi : E \rightarrow B$ is a smooth bundle of closed oriented $2n$ -dimensional manifolds and B is also closed, of dimension p say, then for each $\kappa \in H^p(\Omega_0^\infty \text{MTSO}(2n))$ we get a number

$$K(\kappa) = \int_B \overline{\alpha}_\pi^*(\kappa) \in \mathbb{Z}$$

which we call the *characteristic number* of the bundle associated to the class κ . The process $\kappa \mapsto K(\kappa)$ is additive and gives a homomorphism

$$K : H^p(\Omega_0^\infty \text{MTSO}(2n)) \longrightarrow \mathbb{Z}.$$

For $d = 2n$ and each sequence $C = (c_1, \dots, c_r)$ of elements in \mathcal{B} we get a class $\kappa_C = \kappa_{c_1} \dots \kappa_{c_r} \in H^*(\Omega_0^\infty \text{MTSO}(2n))$, and the homomorphism K is uniquely determined by the characteristic numbers

$$K_C = K(\kappa_C),$$

as $C = (c_1, \dots, c_r)$ runs through sequences of elements of B with $p = \sum(|c_i| - 2n)$.

Theorem 1.1 can be combined with Thom's theorem that any non-zero cohomology class in $H^p(B; \mathbb{Q})$ is detected on some smooth p -manifold $M \rightarrow B$ to deduce that for any non-trivial $C = (c_1, \dots, c_r)$ with $p = \sum(|c_i| - 2n)$ there exists a smooth bundle $\pi : E \rightarrow B$ where B is a closed p -manifold, such that $K_C \neq 0$, and we may even prescribe the bordism class of its fibre. It is a natural question to ask to what extent it is possible to find bundles $\pi : E \rightarrow B$ with prescribed values of all characteristic numbers, perhaps also prescribing the bordism classes $[F]$, $[E]$ and $[B]$ of its fibres, total space, and base space. In this form the question is extremely difficult, and involves the integral homology $H^*(\Omega_0^\infty \text{MTSO}(d))$ which is unknown for $d > 2$ (and very complicated for $d = 2$, cf. [Gal04]). However, if we only ask for realising values of $[E]$, $[B]$ and the numbers K_C up to multiplication by the same positive integer, we can give a complete answer. In other words, for a bundle $\pi : E \rightarrow B$ with $2n$ -dimensional fibres which are all in the same bordism class $[F]$ and p -dimensional base, the bordism classes $[E]$ and $[B]$ and the characteristic numbers determine a point in the projectivisation of the rational vector space

$$(\Omega_{2n+p}^{SO} \oplus \Omega_p^{SO} \oplus \text{Hom}(H^p(\Omega_0^\infty \text{MTSO}(2n)), \mathbb{Z})) \otimes \mathbb{Q}.$$

We prove that the set of points which can be realised by bundles is a rational subvariety cut out by the linear equations (1.2) described below. (The subvariety depends on the bordism class $[F]$ of the fibres of π .)

Our interest in the question of (in)dependence of characteristic number of bundles from bordism classes of fibre, total space and base space came from work of Church, Farb and Thibault: they proved in [CFT12] that certain characteristic numbers of surface bundles depend only on the characteristic numbers of its total space. That work was generalised to higher dimensions by Church, Crossley and Giansiracusa: in [CCG12] they completely classified characteristic numbers of bundles of oriented d -manifolds which depend only on the oriented bordism class of its total space (i.e. not on the base B or the map π). For $d = 2n$, their equations expressing a characteristic number of a bundle as a characteristic number of its total space will be contained in our equations. We shall answer the related question of characteristic numbers of bundles which depend only on the bordism classes of the total space E and the base B (but still not the map $\pi : E \rightarrow B$; in fact, the Church–Crossley–Giansiracusa equations are precisely those of our equations which

do not involve the base and the fibre). We also address the corresponding realisation question: Any set of solutions to our equations is, up to scaling by positive integers, realised by a bundle.

Suggested by the notion of “near-primitive elements” of [CCG12] we make the following definition, which we will use to formulate our equations.

Definition 1.2. Let $d \geq 2$ and $\rho : H^*(BSO; \mathbb{Q}) \rightarrow H^*(BSO(d); \mathbb{Q})$ be the restriction map. We say a class $x \in H^{p+d}(BSO; \mathbb{Q})$ is *almost primitive of order d* if the coproduct $\Delta(x)$ can be written

$$(1.1) \quad \Delta(x) = 1 \otimes x + \sum x_j^p \otimes x_j^d + \sum a_i \otimes b_i$$

where x_j^r has degree r and each b_i either has degree $< d$ or lies in $\text{Ker}(\rho)$. In other words, x is almost primitive of order d when it is sent to $1 \otimes \rho(x)$ under the composition

$$H^*(BSO) \xrightarrow{\Delta} H^*(BSO) \otimes H^*(BSO) \xrightarrow{\text{Id} \otimes \text{proj} \circ \rho} H^*(BSO) \otimes H^{*\geq d+1}(BSO(d)).$$

We write $AP^*(d) \subset H^*(BSO; \mathbb{Q})$ for the vector subspace of such elements.

Explicitly, the space of almost primitive elements is described by the following proposition. Let $\text{ph}_i \in H^{4i}(BSO; \mathbb{Q})$ denote the i th Pontryagin character class (which is primitive for the coproduct Δ), so $H^*(BSO; \mathbb{Q}) \cong \mathbb{Q}[\text{ph}_1, \text{ph}_2, \dots]$.

Proposition 1.3. *For $d \geq 2$, the vector subspace $AP^*(d) \subset \mathbb{Q}[\text{ph}_1, \text{ph}_2, \dots]$ is spanned by those monomials in the ph_i 's having the property that every proper factor has degree $\leq d$. (In particular, the ph_i themselves are almost primitive of any degree.)*

Using the notion of almost-primitive elements, we now describe the linear equations relating $[E]$, $[B]$ and the characteristic numbers of a bundle $\pi : E \rightarrow B$. Suppose B has dimension p and E has dimension $p + 2n$, let $x \in AP^{p+2n}(2n) \subset H^{p+2n}(BSO; \mathbb{Q})$, and $\rho(x) \in H^*(BSO(2n); \mathbb{Q})$ denote its restriction. Using (1.1), the bundle isomorphism $TE \cong \pi^*(TB) \oplus T_\pi E$ gives

$$x(TE) = \rho(x)(T_\pi E) + \sum_j \pi^*(x_j^p(TB)) \cdot x_j^{2n}(T_\pi E) + \sum_i \pi^*(a_i(TB)) \cdot b_i(T_\pi E)$$

where $b_i(T_\pi E)$ is a characteristic class of $T_\pi E$ of degree $< 2n$. Applying $\pi_!$ we obtain the equation

$$\pi_!(x(TE)) = \kappa_{\rho(x)}(\pi) + \sum_j x_j^p(TB) \cdot \int_F x_j^{2n}(TF) \in H^p(B; \mathbb{Q}),$$

and so by integrating over B ,

$$\int_E x(TE) = \int_B \kappa_{\rho(x)}(\pi) + \sum_j \left(\int_B x_j^p(TB) \right) \cdot \left(\int_F x_j^{2n}(TF) \right).$$

The first term on the right hand side is a characteristic number of the bundle π . If we write characteristic numbers of closed manifolds as $\langle x, [E] \rangle = \int_E x(TE)$ and similarly for B and F , the equation can be written as

$$(1.2) \quad \langle x, [E] \rangle = K(\kappa_{\rho(x)}) + \sum_j \langle x_j^p, [B] \rangle \cdot \langle x_j^{2n}, [F] \rangle.$$

For a fixed fibre F , we view (1.2) as a set of linear equations between the bordism classes $[E]$ and $[B]$ and the characteristic numbers $K(\kappa_{\rho(x)})$, one equation for each element x in a basis for the vector space $AP^{p+2n}(2n)$. Our second result is that any formal solution to these relations may be realised by a fibre bundle, up to multiplying by a positive integer.

Theorem 1.4. *Let $2n \in \{2, 6, 8, 10, \dots\}$ and $p > 0$. Fix*

- (i) *a class $f \in \Omega_{2n}^{SO}$,*
- (ii) *a class $e \in \Omega_{2n+p}^{SO}$,*
- (iii) *a class $b \in \Omega_p^{SO}$,*
- (iv) *a homomorphism $K : H^p(\Omega_0^\infty \mathbf{MTSO}(2n)) \rightarrow \mathbb{Z}$.*

Suppose that for each $x \in AP^{2n+p}(2n)$, with coproduct as in (1.1), we have

$$(1.3) \quad \langle x, e \rangle = K(\kappa_{\rho(x)}) + \sum_j \langle x_j^p, b \rangle \cdot \langle x_j^{2n}, f \rangle.$$

Then there exists an integer $N > 0$ and a fibre bundle $\pi : E \rightarrow B$ satisfying condition (iii) in Theorem 1.1, with $[E] = Ne$, $[B] = Nb$, and $\int_B \kappa_C(\pi) = NK(\kappa_C)$ for all sequences $C = (c_1, \dots, c_r)$ of elements of \mathcal{B} with $p = \sum(|c_i| - 2n)$, and with fibres in the bordism class f .

If we do not care about the characteristic numbers $K(\kappa_C)$, we can pick them so $K(\kappa_{\rho(x)})$ satisfies the equations. We get the following corollary.

Corollary 1.5. *Let $2n \in \{2, 6, 8, 10, \dots\}$ and $p > 0$. Fix classes $f, e, b \in \Omega_*^{SO}$ with degrees $2n$, $2n + p$, and p respectively. Then there exists an integer $N > 0$ and a fibre bundle $\pi : E \rightarrow B$, satisfying condition (iii) of Theorem 1.1, such that $[E] = Ne$ and $[B] = Nb$, and such that the fibres of π are in the bordism class f . \square*

Remark 1.6. Condition (iii) in the conclusion of Theorem 1.1 implies that the fibres of π satisfy that the Gauss map to BSO is n -connected. By work of Kreck, such manifolds have very strong rigidity properties. Indeed, by [Kre99, Theorem C], two such are oriented diffeomorphic if and only if they are oriented bordant and have the same Euler characteristic, provided the Euler characteristic is sufficiently large.

Therefore, the statements in Theorem 1.1, Theorem 1.4 and Corollary 1.5 that the fibres of the bundle $\pi : E \rightarrow B$ are in the prescribed bordism class $f \in \Omega_{2n}^{SO}$ could as well have been stated as follows: given a closed $2n$ -dimensional manifold F whose Gauss map $F \rightarrow BSO$ is n -connected, there is a $g \gg 0$ such that all fibres of π are diffeomorphic to $F \# g(S^n \times S^n)$.

2. PROOF OF THEOREM 1.1

Let the data $f \in \Omega_{2n}^{SO}$ and $0 \neq c \in H^*(\Omega_0^\infty \mathbf{MTSO}(2n); k)$ be given.

Lemma 2.1. *There is a manifold F in the bordism class f having the property that the (normal) Gauss map $\nu_F : F \rightarrow BSO$ is n -connected.*

Proof. This may be found in e.g. [Kre84, Theorem 7]. We only require the result for even-dimensional manifolds, which is much easier, so we give it here.

For each $x \in \pi_k(BSO)$ with $k \leq n$, we can find a lift to $\hat{x} \in \pi_k(BSO(2n-k+1))$, i.e. a vector bundle $V_x^{2n-k+1} \rightarrow S^k$ representing this class. The sphere bundle of V , $S(V_x)$, is a $2n$ -manifold and by construction has the property that its normal Gauss map $\nu_{S(V_x)} : S(V_x) \rightarrow BSO$ hits the class x on π_k .

If we choose any $F'' \in f$ we may take connected-sum with $S(V_x)$ for generators x of $\pi_{* \leq n}(BSO)$ to obtain a manifold F' such that $\nu_{F'}$ is surjective on homotopy groups in degrees $* \leq n$. Furthermore, $[F''] = f$ as the manifolds $S(V_x)$ are nullbordant (they bound the associated disc-bundle). Now, we perform surgery on F' along spheres generating the kernels of $\pi_k(F') \rightarrow \pi_k(BSO)$ for $k \leq n-1$. The manifold we end up with, F , has ν_F n -connected. \square

Proof of Theorem 1.1. Let F be a manifold satisfying the conclusion of Lemma 2.1, and let $h : F \rightarrow [0, 2n]$ be a self-indexing Morse function. Let $P = h^{-1}(n - 1/2)$ and define

$$\begin{aligned} W &= h^{-1}([0, n - \frac{1}{2}]), \\ A &= h^{-1}([n - \frac{1}{2}, n + \frac{1}{2}]), \\ M &= h^{-1}([n - \frac{1}{2}, 2n]). \end{aligned}$$

Then A is a bordism from P to $h^{-1}(n + 1/2)$, and we let \overline{A} denote the bordism in the other direction with opposite orientation. Then define bordisms from P to itself by

$$\begin{aligned} K_0 &= A \cup_{h^{-1}(n+1/2)} \overline{A} \\ K_1 &= ([0, 1] \times P) \# (S^n \times S^n) \\ K &= K_0 \cup_P K_1. \end{aligned}$$

It is easy to see that K_0 and K_1 commute, in the sense that there is a diffeomorphism $K_0 \cup_P K_1 \cong K_1 \cup_P K_0$, relative to the boundary.

By construction, this data (W, K) satisfies the assumptions explained in [GRW12, Remark 1.11]¹, and furthermore the n th stage of the Moore–Postnikov factorisation of the tangential Gauss map $K \rightarrow BO(2n)$ is $BSO(2n) \rightarrow BO(2n)$. Thus the Pontryagin–Thom construction gives a map

$$(2.1) \quad \text{hocolim}_{g \rightarrow \infty} BDiff(W \cup gK, \partial) \longrightarrow \Omega_0^\infty \mathbf{MTSO}(2n)$$

and it follows from [GRW12, Theorem 1.8] that this map induces an isomorphism in integral homology and hence in any generalised homology theory.

Now let $c \neq 0 \in H^p(\Omega_0^\infty \mathbf{MTSO}(2n); k)$. Since $H_*(\Omega_0^\infty \mathbf{MTSO}(2n))$ is finitely generated in each degree, $H^*(\Omega_0^\infty \mathbf{MTSO}(2n); k)$ is the direct limit of cohomology with coefficients in finitely generated subgroups of k , so c is in the image of cohomology with coefficients in some finitely generated subgroup $k' \subset k$. Then there exists a class $x \in H_p(\Omega_0^\infty \mathbf{MTSO}(2n); \text{Hom}(k', \mathbb{Q}/\mathbb{Z}))$ with $\langle x, c \rangle \neq 0 \in \mathbb{Q}/\mathbb{Z}$. We proved above that x is in the image from homology of some

$$BDiff(W \cup_P gK, \partial) \simeq BDiff(W \cup_P gK \cup_P M, M),$$

and since any homology class is supported on a finite subcomplex of a CW approximation, there exists a manifold B and a map $f : B \rightarrow BDiff(W \cup_P gK \cup_P M, M)$ such that x is in the image of f_* . It follows that

$$f^*(c) \neq 0 \in H^p(B; k)$$

and hence that the map f classifies a bundle $\pi : E \rightarrow B$ with all the required properties. \square

3. PROOF OF THEOREM 1.4

The strategy of the proof will be to first solve a bordism version of the problem of Theorem 1.4, and then appeal to the results of [GRW12] to upgrade this bordism solution to a fibre bundle solution. The bordism version does not have the restriction $d = 2n \neq 4$, it holds in any dimension $d \geq 2$.

¹The statement of [GRW12, Theorem 1.8] says $2n > 4$, but it also holds for $2n = 2$ by the Madsen–Weiss theorem [MW07], and for $2n = 0$ by the Barratt–Priddy theorem [BP72].

Recall that the classes κ_c are defined universally in $H^{|c|-d}(\Omega^\infty \mathbf{MTSO}(d))$, and that the map

$$\mathbb{Q}[\kappa_c \mid c \in \mathcal{C}] \longrightarrow H^*(\Omega^\infty \mathbf{MTSO}(d); \mathbb{Q})^{\pi_0} \cong H^*(\Omega_0^\infty \mathbf{MTSO}(d); \mathbb{Q})$$

to the subring of $\pi_0(\Omega^\infty \mathbf{MTSO}(d))$ -invariant classes is an isomorphism, where \mathcal{C} is the set of monomials of degree $> d$ in $H^*(BSO(d); \mathbb{Q})$ in the variables $p_1, \dots, p_{\lfloor d/2 \rfloor}$ if d is odd or $p_1, \dots, p_{\lfloor d/2 \rfloor - 1}, e$ if d is even.

By Pontryagin–Thom theory, the infinite loop space $\Omega^\infty \mathbf{MTSO}(d)$ classifies “formal oriented d -dimensional fibre bundles”. That is, for any smooth manifold B the Pontryagin–Thom construction gives a natural bijection

$$[B, \Omega^\infty \mathbf{MTSO}(d)] \longleftrightarrow \left\{ \begin{array}{l} \pi : E \rightarrow B \text{ smooth proper map,} \\ V \rightarrow E \text{ } d\text{-dimensional oriented vector bundle,} \\ \varphi : TE \xrightarrow{\sim_s} V \oplus \pi^*(TB) \text{ stable isomorphism.} \end{array} \right\}$$

between homotopy classes of maps and cobordism classes over B of formal fibre bundles. The Miller–Morita–Mumford classes can be defined in this theory: to $c \in H^{p+d}(BSO(d))$ and $[\pi, E, B, V, \varphi] \in [B, \Omega^\infty \mathbf{MTSO}(d)]$ we let

$$\kappa_c = \pi_!(c(V)) \in H^p(B),$$

where the Gysin map $\pi_!$ is defined using Poincaré duality in E and B . Under the bijection, associating the map $\alpha_\pi : B \rightarrow \Omega^\infty \mathbf{MTSO}(d)$ to a bundle $\pi : E \rightarrow B$ with oriented d -dimensional fibres corresponds to forgetting that π is a bundle, remembering only the stable isomorphism $TE \xrightarrow{\sim_s} \pi^*(TB) \oplus T_\pi E$ induced by the differential of π . For the universal bundle with fibre F we get the map

$$B\mathrm{Diff}(F) \longrightarrow \Omega^\infty \mathbf{MTSO}(d).$$

Writing $\Omega_*^{SO}(-)$ for oriented bordism theory, there is a natural bijection

$$\Omega_p^{SO}(\Omega^\infty \mathbf{MTSO}(d)) \longleftrightarrow \left\{ \begin{array}{l} \pi : E^{p+d} \rightarrow B^k \text{ smooth proper map,} \\ V^d \rightarrow E \text{ oriented vector bundle,} \\ \varphi : TE \xrightarrow{\sim_s} V \oplus \pi^*(TB) \text{ stable isomorphism.} \end{array} \right\}$$

to the set of cobordism classes of such data, where B may also change by a cobordism. The data relevant for Theorem 1.4 can all be extracted from this group: the bordism classes $[F]$, $[E]$, and $[B]$, and the characteristic numbers K_C . Firstly, we can define characteristic numbers K_C

$$K_C(\pi, E, B, V) = \int_B \kappa_{c_1}(\pi, V) \cdots \kappa_{c_n}(\pi, V) \in \mathbb{Z}.$$

It is easy to check that these characteristic numbers are invariants of the cobordism class. There is an obvious map $\Omega_p^{SO}(\Omega^\infty \mathbf{MTSO}(d)) \rightarrow \Omega_p^{SO}(*)$ which sends $[\pi, E, B, V, \varphi]$ to $[B]$. Similarly, sending a bordism class $[\pi, E, B, V, \varphi]$ to the bordism class $[E] \in \Omega_{p+d}^{SO}$ gives a well defined homomorphism

$$\Omega_p^{SO}(\Omega^\infty \mathbf{MTSO}(d)) \rightarrow \Omega_{p+d}^{SO}.$$

We point out that this homomorphism is *not* invariant under translation to different path components, and in this section it often is better *not* to translate back to the path component of the basepoint. In fact, the bordism class of the fibre $[F]$ corresponds to the group of path components of $\Omega^\infty \mathbf{MTSO}(d)$. There is a stabilisation map $\mathbf{MTSO}(d) \rightarrow \Sigma^{-d} \mathbf{MSO}$ which induces a surjection

$$\pi_0(\Omega^\infty \mathbf{MTSO}(d)) \longrightarrow \Omega_d^{SO}.$$

If $f \in \Omega_d^{SO}$ is a bordism class we write $\Omega_{(f)}^\infty \mathbf{MTSO}(d)$ for the collection of path components which go to f under this map. If $\pi : E \rightarrow B$ is a bundle with fibres in the bordism class f , then the corresponding map $\alpha_\pi : B \rightarrow \Omega^\infty \mathbf{MTSO}(d)$ has image in $\Omega_{(f)}^\infty \mathbf{MTSO}(d)$.

The cobordism version of Theorem 1.4 is the problem of finding a class

$$[\pi, E, B, V, \varphi] \in \Omega_{d+p}^{SO}(\Omega_{(f)}^\infty \mathbf{MTSO}(d))$$

with $[E] = e \in \Omega_{d+p}^{SO}$, $[B] = b \in \Omega_p^{SO}$ and which maps to a given functional $K \in \text{Hom}(H^p(\Omega_0^\infty \mathbf{MTSO}(d)), \mathbb{Z}) \subset \text{Hom}(H^p(\Omega_0^\infty \mathbf{MTSO}(d)), \mathbb{Q})$ under

$$\begin{aligned} \Omega_p^{SO}(\Omega^\infty \mathbf{MTSO}(d)) &\longrightarrow H_p(\Omega^\infty \mathbf{MTSO}(d); \mathbb{Q}) \longrightarrow H_p(\Omega_0^\infty \mathbf{MTSO}(d); \mathbb{Q}) \\ &= \text{Hom}(H^p(\Omega_0^\infty \mathbf{MTSO}(d)), \mathbb{Q}), \end{aligned}$$

where the last map is induced by translating to the basepoint component.

Proposition 3.1. *Let $d \geq 2$ and $p > 0$ and fix*

- (i) *a class $f \in \Omega_d^{SO}$,*
- (ii) *a class $e \in \Omega_{d+p}^{SO}$,*
- (iii) *a class $b \in \Omega_p^{SO}$,*
- (iv) *a homomorphism $K : H^p(\Omega_0^\infty \mathbf{MTSO}(d)) \rightarrow \mathbb{Z}$.*

Suppose that these data satisfy equation (1.3) for each $x \in AP^{d+p}(d)$. Then there exists an integer $N > 0$ and a class

$$[\pi, E, B, V, \varphi] \in \Omega_p^{SO}(\Omega_{(f)}^\infty \mathbf{MTSO}(d)_+)$$

such that $[E] = Ne$, $[B] = Nb$ and $\int_B \kappa_C(\pi, V) = NK_C$ for all sequence $C = (c_1, \dots, c_r)$ of elements of \mathcal{B} with $p = \sum(|c_i| - 2n)$.

We first show how to deduce Theorem 1.4 from Proposition 3.1.

Proof of Theorem 1.4. This is essentially the same proof as that of Theorem 1.4. With notation as in the proof of that theorem, the bordism class provided by proposition 3.1 is in the image of some element of

$$\Omega_p^{SO}(\text{BDiff}(W \cup_P gK \cup_P M, M))$$

under the map in oriented bordism induced by the Pontryagin–Thom map

$$\text{BDiff}(W \cup_P gK \cup_P M, M) \longrightarrow \Omega_{(f)}^\infty \mathbf{MTSO}(2n).$$

An element $\Omega_p^{SO}(\text{BDiff}(W \cup_P gK \cup_P M, M))$ mapping to $[\pi, E, B, V, \varphi]$, is given by a bundle $\pi : E \rightarrow B$ with fibre $W \cup_P gK \cup_P M$. \square

Before proving Proposition 3.1, we first show it suffices to establish the case when $b = 0$ and $f = 0$.

Lemma 3.2. *If Proposition 3.1 is true when $b = 0$ and $f = 0$ then it is true for arbitrary b and f .*

Proof. Let (f, e, b, K_C) be data we wish to realise. The associated data $(0, e - b \cdot f, 0, K_C)$ still satisfies the compatibility conditions so by assumption we can realise it by some $[\pi', B', E', V', \varphi']$.

Then we pick some $B'' \in b$ and set $B = B \amalg B''$. Then $B \in b$, and the composition $\pi : E' \rightarrow B' \hookrightarrow B$ realises the data $(0, e - b \cdot f, b, K_C)$. Then we pick $F \in f$, set $E = E' \amalg (B \times F)$, and extend π by the projection $B \times F \rightarrow B$. Since the trivial bundle $B \times F \rightarrow B$ has vanishing characteristic numbers, this realises (f, e, b, K_C) . \square

It remains to prove Proposition 3.1 in the case $b = f = 0$. When $f = 0$ the space $\Omega_{(0)}^\infty \mathbf{MTSO}(d)$ still consists of many components, and we shall in fact show something stronger, that we may find an element in $\Omega_p^{SO}(\Omega_0^\infty \mathbf{MTSO}(d))$, the bordism of the basepoint component. To satisfy the requirement $b = 0$ we are asking for an element of reduced bordism $\Omega_p^{SO}(\Omega_0^\infty \mathbf{MTSO}(d), *)$.

Let us write \mathbf{MSO} for the spectrum representing oriented bordism, and $u : \mathbf{MSO} \rightarrow \mathbf{H}\mathbb{Z}$ for the map of spectra representing its Thom class. Let $\overline{\mathbf{MSO}}$ denote the homotopy fibre of u , and $\overline{\Omega}_*^{SO}(-)$ be the associated homology theory. Let us write $\mathbf{mtso}(d)$ for the 0-connected cover of $\mathbf{MTSO}(d)$. We will establish a commutative diagram

$$\begin{array}{ccccc}
\overline{\Omega}_p^{SO}(\Omega_0^\infty \mathbf{MTSO}(d), *) \otimes \mathbb{Q} & \xrightarrow{\sigma} & \overline{\Omega}_p^{SO}(\mathbf{mtso}(d)) \otimes \mathbb{Q} & = & \overline{\Omega}_p^{SO}(\mathbf{mtso}(d)) \otimes \mathbb{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_p^{SO}(\Omega_0^\infty \mathbf{MTSO}(d), *) \otimes \mathbb{Q} & \xrightarrow{\sigma} & \Omega_p^{SO}(\mathbf{mtso}(d)) \otimes \mathbb{Q} & \xrightarrow{\mathcal{E}} & \Omega_{p+d}^{SO} \otimes \mathbb{Q} \\
\downarrow & & \downarrow & \textcircled{1} & \downarrow \\
H_p(\Omega_0^\infty \mathbf{MTSO}(d), *; \mathbb{Q}) & \xrightarrow{\sigma} & H_p(\mathbf{mtso}(d); \mathbb{Q}) & \longrightarrow & AP^{p+d}(d)^\vee
\end{array}$$

in which all columns are exact at the second row, and maps are injective or surjective as indicated. Proposition 3.1 for the case $b = f = 0$ concerns elements of $\Omega_p^{SO}(\Omega_0^\infty \mathbf{MTSO}(d), *) \otimes \mathbb{Q}$ with specified image in $H_p(\Omega_0^\infty \mathbf{MTSO}(d), *; \mathbb{Q})$ and Ω_{p+d}^{SO} , and the proof follows by an easy diagram chase once the diagram is established.

The maps σ are realised by maps of spectra

$$\sigma : \Sigma^\infty \Omega_0^\infty \mathbf{MTSO}(d) \longrightarrow \mathbf{mtso}(d)$$

coming from the $\Sigma^\infty - \Omega^\infty$ adjunction, so the left-hand half of the diagram commutes. The map σ is surjective on any rationalised homology theory.

The map \mathcal{E} is realised by

$$\mathbf{MSO} \wedge \mathbf{mtso}(d) \rightarrow \mathbf{MSO} \wedge \mathbf{MTSO}(d) \rightarrow \mathbf{MSO} \wedge \Sigma^{-d} \mathbf{MSO} \xrightarrow{\mu} \Sigma^{-d} \mathbf{MSO}$$

where the first map comes from $\mathbf{mtso}(d)$ being the connective cover of $\mathbf{MTSO}(d)$, the second comes from the stabilisation and the last map comes from the multiplicative structure of \mathbf{MSO} . On rational cohomology we have a square

$$\begin{array}{ccc}
H^p(\mathbf{MSO} \wedge \mathbf{mtso}(d); \mathbb{Q}) & \xleftarrow{\mathcal{E}^*} & H^p(\Sigma^{-d} \mathbf{MSO}; \mathbb{Q}) \\
\uparrow & & \uparrow \text{Thom} \cong \\
H^p(\mathbf{mtso}(d); \mathbb{Q}) & \xleftarrow[\cong]{\text{Thom}} & H^{p+d}(BSO(d); \mathbb{Q}) \xleftarrow{\rho} H^{p+d}(BSO; \mathbb{Q})
\end{array}$$

which does not commute. Unwinding definitions, we see that the subset $AP^{p+d}(d)$ of $H^{p+d}(BSO; \mathbb{Q})$ is the largest subspace for which the diagram does commute. Dualising we obtain the commutative square $\textcircled{1}$.

Lemma 3.3. *The map $AP^*(d) \hookrightarrow H^*(BSO; \mathbb{Q}) \xrightarrow{\rho} H^*(BSO(d); \mathbb{Q})$ is injective for $d \geq 2$.*

Proof. An element x in the kernel is simultaneously in $AP^*(d)$ and the ideal $(p_{\lfloor d/2 \rfloor + 1}, p_{\lfloor d/2 \rfloor + 2}, \dots) \subset H^*(BSO; \mathbb{Q})$. If x is non-trivial, it must then have degree $|x| \geq |p_{\lfloor d/2 \rfloor + 1}| = 4\lfloor d/2 \rfloor + 4$. However, by the classification of almost primitives in Proposition 1.3, a class which is almost primitive of order d and of degree $> 2 \cdot d$ must be an ordinary primitive, so x must be a multiple of a Pontryagin character class. But no Pontryagin character class lies in the ideal $(p_{\lfloor d/2 \rfloor + 1}, p_{\lfloor d/2 \rfloor + 2}, \dots)$ if $d \geq 2$, as they contain a non-trivial monomial which is a power of p_1 . \square

It follows that the two horizontal maps in the square $\textcircled{1}$ are surjective for $d \geq 2$. We now turn to the exactness of the columns. The first two follows for general

reasons: a cofibration sequence of spectra induces long exact sequences of the corresponding homology theories. It remains to consider the third column.

Lemma 3.4. *The sequence*

$$\overline{\Omega}_p^{SO}(\mathbf{m}t\mathbf{s}\mathbf{o}(d)) \otimes \mathbb{Q} \longrightarrow \Omega_{k+d}^{SO} \otimes \mathbb{Q} \longrightarrow AP^{k+d}(d)^\vee \longrightarrow 0$$

is exact.

Proof. Dualising, we must show that if a class $x \in H^p(\Sigma^{-d}\mathbf{MSO}; \mathbb{Q})$ is zero under the composition

$$\overline{\mathbf{MSO}} \wedge \mathbf{m}t\mathbf{s}\mathbf{o}(d) \longrightarrow \mathbf{MSO} \wedge \Sigma^{-d}\mathbf{MSO} \longrightarrow \Sigma^{-d}\mathbf{MSO},$$

then considered as a class in $H^{k+d}(BSO; \mathbb{Q})$ it is almost primitive of order d . But on cohomology, after applying the Thom isomorphism, this composition is

$$H^*(BSO) \xrightarrow{\Delta} H^*(BSO) \otimes H^*(BSO) \xrightarrow{\text{proj}} H^{*\geq 1}(BSO) \otimes H^{*\geq d+1}(BSO(d))$$

so if x is sent to zero, it is *by definition* almost primitive of order d . \square

This finishes the proof of Proposition 3.1.

4. PROOF OF PROPOSITION 1.3

Recall that we write $\text{ph}_i \in H^{4i}(BSO; \mathbb{Q})$ for the i th Pontryagin character class, so that $H^*(BSO; \mathbb{Q}) \cong \mathbb{Q}[\text{ph}_1, \text{ph}_2, \dots]$. We wish to show that the vector subspace $AP^*(d) \subset \mathbb{Q}[\text{ph}_1, \text{ph}_2, \dots]$ is spanned by the monomials $\text{ph}_{a_1} \cdots \text{ph}_{a_k}$ such that every proper submonomial $\text{ph}_{a_1} \cdots \text{ph}_{a_{j-1}} \cdot \text{ph}_{a_{j+1}} \cdots \text{ph}_{a_k}$ has degree $\leq d$. It is clear that all such monomials are almost primitive of degree d . For a sequence $I = (i_1, i_2, \dots)$ of non-negative integers with only finitely many non-zero terms, we shall write

$$\text{ph}^I = \prod_j \text{ph}_j^{i_j}.$$

In this notation $\text{ph}^I \cdot \text{ph}^J = \text{ph}^{I+J}$. The $\text{ph}^I \in H^*(BSO; \mathbb{Q})$ form a basis, and we shall always express elements of $H^*(BSO; \mathbb{Q})$ in this basis.

The following argument is similar to [CCG12, §6].

Claim 4.1. Every ph^J occurring as a proper factor of a monomial of $x \in AP^*(d)$ either has $|\text{ph}^J| \leq d$ or $|\text{ph}^J| > 4 \cdot \lfloor d/2 \rfloor$.

Proof. If $\text{ph}^I \cdot \text{ph}^J$ occurs as a monomial in x with non-zero coefficient, we may write

$$x = \text{ph}^I \cdot \left(\sum_{J'} \alpha_{J'} \text{ph}^{J'} \right) + y,$$

where y is a linear combination of monomials not divisible by ph^I , and the sum is over multiindices J' with $|\text{ph}^{J'}| = |\text{ph}^J|$ (so $\alpha_{J'} \neq 0$). Then

$$\Delta(x) = \text{ph}^I \otimes \left(\sum_{J'} \alpha_{J'} \binom{I+J'}{I} \text{ph}^{J'} \right) + z,$$

where we write $\binom{K}{I} = \prod_n \binom{k_n}{i_n}$ and z is a linear combination of terms not of the form $\text{ph}^I \otimes \text{ph}^{J''}$. As x is almost primitive, we deduce that the element

$$\sum \alpha_{J'} \binom{I+J'}{I} \text{ph}^{J'} \in H^*(BSO; \mathbb{Q}),$$

which is non-zero since $\alpha_J \binom{I+J}{I} \neq 0$, either has degree $\leq d$ or is in the kernel of the map $\rho : H^*(BSO; \mathbb{Q}) \rightarrow H^*(BSO(d); \mathbb{Q})$. Since this kernel vanishes in degrees $\leq 4 \lfloor d/2 \rfloor$ we have proved the claim. \square

Proof of Proposition 1.3. To prove the proposition, suppose for contradiction that $x \in AP^*(d)$ contains a monomial of the form $\text{ph}^I \cdot \text{ph}^J$ with $|\text{ph}^I| > 0$ and $|\text{ph}^J| > d$. We may assume that $|\text{ph}^J|$ is minimal with this property. The claim above shows that in fact $|\text{ph}^J|/4 > \lfloor d/2 \rfloor$ so in particular $|\text{ph}^J| > 2d$. It follows that $\text{ph}^J = \text{ph}_j$ for some j , since any proper factor of ph^J must have degree $\leq d$ by minimality of J and hence can be combined to a proper factor of degree $\in (d, 2d]$ contradicting the statement of the claim.

We have proved that x contains a monomial of the form $\text{ph}^I \cdot \text{ph}_j$ with $|\text{ph}_j| > 2d$. As in the proof of the claim, we may write

$$x = \text{ph}_j \cdot \left(\sum_{I'} \alpha_{I'} \text{ph}^{I'} \right) + y,$$

where y is a sum of monomials not divisible by ph_j , and $\alpha_I \neq 0$. Then

$$\Delta(x) = \left(\sum_{I'} \alpha_{I'} (1 + i'_j) \text{ph}^{I'} \right) \otimes \text{ph}_j + z$$

where z is a linear combination of terms not of the form $\text{ph}^{I'} \otimes \text{ph}_j$. As $|\text{ph}_j| > d$ and $x \in AP^*(d)$, it follows that ph_j is in the kernel of $H^*(BSO; \mathbb{Q}) \rightarrow H^*(BSO(d); \mathbb{Q})$. But this is a contradiction since $d \geq 2$, and ph_j is non-zero even in $H^*(BSO(2); \mathbb{Q})$. \square

Remark 4.2. The *near-primitive elements of order d* discussed by Church, Crossley and Giansiracusa are the almost primitive elements of order d for which in addition $\sum_j x_j^p \otimes x_j^q = 0$. We write $NP^*(d) \subset AP^*(d)$ for that subspace. In [CCG12, Theorem A] they give an explicit description of the near primitive elements, which combined with Proposition 1.3 shows that $AP^*(d) = NP^*(d+1)$.

REFERENCES

- [BP72] Michael Barratt and Stewart Priddy, *On the homology of non-connected monoids and their associated groups*, Comment. Math. Helv. **47** (1972), 1–14.
- [CCG12] Thomas Church, Martin Crossley, and Jeffrey Giansiracusa, *Invariance properties of Miller–Morita–Mumford characteristic numbers of fibre bundles*, The Quarterly Journal of Mathematics (2012).
- [CFT12] Thomas Church, Benson Farb, and Matthew Thibault, *On the geometric nature of characteristic classes of surface bundles*, J. Topol. **5** (2012), no. 3, 575–592.
- [Ebe09] Johannes Ebert, *A vanishing theorem for characteristic classes of odd-dimensional manifold bundles*, arXiv:0902.4719, to appear in *Crelle’s Journal*, 2009.
- [Ebe11] _____, *Algebraic independence of generalized MMM-classes*, Algebr. Geom. Topol. **11** (2011), no. 1, 69–105.
- [Gal04] Søren Galatius, *Mod p homology of the stable mapping class group*, Topology **43** (2004), no. 5, 1105–1132.
- [GRW12] Søren Galatius and Oscar Randal-Williams, *Stable moduli spaces of high dimensional manifolds*, arXiv:1201.3527, 2012.
- [Kre84] M. Kreck, *Manifolds with unique differentiable structure*, Topology **23** (1984), no. 2, 219–232.
- [Kre99] Matthias Kreck, *Surgery and duality*, Ann. of Math. (2) **149** (1999), no. 3, 707–754.
- [MW07] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941.

E-mail address: galatius@stanford.edu

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD CA, 94305

E-mail address: o.randal-williams@dpmms.cam.ac.uk

DPMMS, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK